Lower bound for transport for passive advection

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(Received 11 April 1994)

The lower bound for the passively advected flux is studied. It is completely rigorous and turns out to be close to, or of the order of magnitude of, the true value. The lower bound is obtained through the variational principle which incorporates the physical constraints—i.e., the conservation laws of the flux and the energy. It can be argued that the same method can be applied to a variety of problems with physical implications.

PACS number(s): 47.27.−i, 52.25.Fi, 52.35.Ra

INTRODUCTION

In this paper, we explore the lower bounds for transport for passive advection. The scope of this work is pedagogical. However, the method adopted in the present work can be applied to a wider class of problems of turbulence.

Turbulence is a quite general phenomenon observed in fluids and plasmas. It is unpredictable, irregular, or random and enhances the transport. It is unpredictable because a small uncertainty about the initial conditions increases so fast that it is difficult to predict the behavior at later times. One thus tries to model the turbulence and to solve it as rigorously as possible. However, mainly due to the inherent nonlinearity involved in the problem, it is hard to study turbulence problems. Let us consider a typical problem in which fluid is advected by a velocity field $\boldsymbol{u}$ inside a box. One is interested in the statistically steady state where mean quantities are independent of time and the correlations between two quantities at different times depend only on the time difference. For simplicity, consider the case where the mean velocity is zero everywhere. In a turbulent state, the flux has a contribution from advection due to the flow as well as the conduction (diffusion) due to the dissipation. One may want to compute the flux $\boldsymbol{\Gamma}(\boldsymbol{x})$ of a field T due to the fluctuations $\boldsymbol{\delta u}$ and δT. [$\boldsymbol{\Gamma}(\boldsymbol{x}) \doteq \langle \boldsymbol{\delta u}(\boldsymbol{x},t)\delta T(\boldsymbol{x},t)\rangle$.] Various techniques can be used to compute or determine information about $\boldsymbol{\Gamma}$. One can do dimensional analysis, numerical simulations on computers [1], or closure approximations employing statistical methods that include the weak turbulence theory [2], the direct-interaction approximation [3], the renormalization-group theory [4,5], and the decimated-amplitude scheme [6]. Each of these methods has its own difficulties and they are well documented in each reference cited. For instance, for the numerical simulations of turbulent flows, a large memory content is required because a great range of spatial scales is involved in the problem. For the closure approximations, their applicability is limited and the assumptions are rarely justifiable.

Although the closure theories exist in their own right and provide dynamical information in many cases, one may need another way of estimating turbulent transport, which may contain only fundamental physics if not all the dynamical details. It is, also, nice to have a benchmark to which one can compare the results of the closure theories. If it is rigorous, so much the better because rigorous results are rare in turbulence research. Howard [7] obtained a rigorous *upper* bound of the flux by formulating a variational principle which employs dynamical constraints satisfied by the physical system. Since the predicted upper bound turns out to follow closely the experimentally observed value, extensive works on thermal convection and on other situations in neutral fluids have followed [8]. For plasma turbulence, the same method is applied to the reversed-field pinches [9]. Since one knows how to obtain reasonably good upper bounds by following Howard [7], it seems natural to consider rigorous lower bounds within the framework of Howard's theory as well. Thus this work is part of a continuing effort to bound the transport with relatively little difficulty: In this work we try to bound the flux from below.

The single most important issue in rigorously bounding the transport is the derivation of the physical constraints. The method adopted in this work follows closely the procedures in the study of upper bounds in [10]. The simplest dynamical constraints are obtained from the conservation laws of the flux and the energy—namely, by balancing the energy dissipation with the energy production due to the interaction between the mean fields and the fluctuations. In a schematic notation, the energy balance in the steady state can be rewritten as follows: $\Gamma = \mathcal{D} + \mathcal{N}$, where Γ is the mean flux one wants to maximize, $\mathcal{D}$ represents the dissipation, and $\mathcal{N}$ is the nonlinear part of the energy production terms. Both $\mathcal{D}$ and $\mathcal{N}$ are positive definite quantities. For the details of the above relation, one is referred to Eq. (5) in the main text. In order to obtain the simplest upper bound for the flux one maximizes Γ obeying the above relation [10]. Of course, one can improve upper bounds by incorporating more physical constraints on top of the above energy relation as shown in [11]. For the *lower* bound, one can use the above relation more usefully by noting that the flux is always larger than both $\mathcal{D}$ and $\mathcal{N}$ because both of them are positive. Thus, if one finds the upper bound of either $\mathcal{D}$ or $\mathcal{N}$ that satisfies the energy balance, it serves as a lower bound for the flux. The variational principle in this context is thus *"Maximize $\mathcal{D}$ or $\mathcal{N}$ satisfying the energy balance."*

LOWER BOUND

In this work, we consider the situation in which a scalar field T is passively advected by a random velocity field u, which depends only on time, in the presence of dissipation in a bounded region. We assume that u is a centered, time-stationary, and Gaussian random variable. The boundary conditions are $T = 1$ at $x = 0$ and $T = 0$ at $x = 1$; the fluctuation δT vanish at the boundaries. Except the passive nature, this model is of a generic type for a wide class of problems. The model equation is of the diffusion-advection type:

$$\frac{\partial T}{\partial t} + u(t)\frac{\partial T}{\partial x} - R^{-1}\frac{\partial^2 T}{\partial x^2} = 0, \tag{1}$$

where R is the "Reynolds number" representing the dissipation. In the present model there are two parameters; one is R and the other is K, the Kubo number, representative of the autocorrelation time. More detailed discusstions about the nature of Eq. (1) can be found in [10]. When K is infinite, the exact solution of Eq. (1) can be readily obtained [10]. For the case of finite K the exact solution has not been found analytically and one has to solve it numerically as in [11].

In order to obtain the lower bound we proceed first by taking the average of Eq. (1) over the ensemble of u. In the steady state, one finds

$$\frac{d}{dx}\langle \delta u \delta T\rangle - R^{-1}\frac{d^2}{dx^2}\langle T\rangle = 0, \tag{2}$$

where the angular brackets mean the ensemble averages. Equation (2) states the conservation of the total flux—namely, the sum of the turbulent flux $\langle \delta u\delta T\rangle$ and the diffusive (classical) flux $-R^{-1}d\langle T\rangle/dx$ is constant. This can be seen by integrating Eq. (2) over x. Upon using the boundary conditions, one can rewrite Eq. (2) to a more convenient form:

$$\frac{d\langle T\rangle}{dx} = R\left[\Gamma(x,0) - \overline{\Gamma}(0) - R^{-1}\right], \tag{3}$$

where $\Gamma(x,\tau) \doteq \langle \delta u(t-\tau)\delta T(x,t)\rangle$ and $\overline{\Gamma}(\tau) \doteq \int_0^1 dx\, \Gamma(x,\tau)$. In order to obtain the relation for the energy balance one finds an equation for the fluctuation δT by subtracting Eq. (2) from Eq. (1), by multiplying the result by $\delta T(x,t)$, and by ensemble averaging. The energy balance equation in the steady state is

$$\Gamma(x,0)\frac{d\langle T\rangle}{dx} + \frac{d}{dx}\left\langle \tfrac{1}{2}\delta u \delta T^2\right\rangle - R^{-1}\left\langle \delta T\frac{\partial^2 \delta T}{\partial x^2}\right\rangle = 0, \tag{4}$$

where "energy" is termed for the variance $\langle \delta T^2/2\rangle$. One may interpret the first term as the production of energy due to the interaction of the mean gradient and the fluctuation, the second as transfer, and the third as dissipation. As is usually the case, one notices that the closure problem arises—that is, the second term in Eq. (4) is of higher order and one needs to express it in terms of the lower-order moments. Rather than attempting to approximate this triplet term, in the bounding theory, one proceeds rigorously by annihilating the higher-order term by employing the boundary conditions after integrating Eq. (4). Upon using Eq. (3) one obtains the relation

$$\overline{\Gamma}(0) = \mathcal{D} + \mathcal{N}, \tag{5}$$

where $\mathcal{N} \doteq R\overline{\Delta\Gamma^2}$, $\mathcal{D} \doteq R^{-1}\overline{\left\langle (\partial\delta T/\partial x)^2\right\rangle}$, $\Delta\Gamma = \Gamma(x,0) - \overline{\Gamma}(0)$, and the bar denotes the integration over x. Now, as in [10], if one is interested in the upper bound, one can formulate a variational principle that maximizes $\overline{\Gamma}(0)$ under the condition of Eq. (5). Also, if one wants to improve the upper bound, one needs to take more physical constraints into the formalism as in [11].

In order to find the lower bound one can utilize Eq. (5) further. That is, by noting that the terms in the right-hand side of Eq. (5) are positive definite, the flux should be larger than both of them. Thus, if one knows the upper bound of either term, one can consider it as the lower bound for the flux. Thus instead of maximizing the flux we are maximizing either $\mathcal{D}$ or $\mathcal{N}$. For example, if we go on to maximize $\mathcal{D}$ under the condition of Eq. (5) (the variational functional is $\mathcal{L} = \mathcal{D} + \lambda[\overline{\Gamma}(0) - \mathcal{D} - \mathcal{N}]$ with the multiplier λ to be determined), then the Euler-Lagrange equation becomes

$$-2R^{-1}\delta T'' + \lambda(u + 2R^{-1}\delta T'' - 2Ru\Delta\Gamma) = 0. \tag{6}$$

After multiplying Eq. (6) by δT and upon using Eq. (5) one finds $\lambda = 2\mathcal{D}/\overline{\Gamma}(0)$. Make a note that there is no time dependence in Eq. (6). This is because the constraint does not have any time information. Effectively, by solving Eq. (6) one obtains the lower bound of the flux in the case of infinite autocorrelation time K. Upon multiplying Eq. (6) one can reduce Eq. (6) to

$$\Gamma'' - \alpha^2\Delta\Gamma = -\alpha^2/2R, \tag{7}$$

where $\alpha^2 = R^2\lambda/(\lambda-1)$. By defining $\Gamma \doteq \left[\overline{\Gamma} + 1/(2R)\right]\hat{\Gamma}$ one finds

$$\hat{\Gamma}(x) = 1 - \phi(x)\ , \tag{8}$$

where $\phi(x) \doteq \cosh[\alpha(x-1/2)]/\cosh(\alpha/2)$. Thus $\overline{\Gamma} = (1-\Phi)/2R\Phi$ and $\Phi = \tanh(\alpha/2)/(\alpha/2)$. After lengthy algebra it is found that for $R = 10$ the lower bound is 0.29 and for $R = 50$ it is 0.32. The result for the case of $R = 10$ is shown in Fig. 1 with the symbol of the filled circle. The true value for the flux when $R = 10$ is 0.31 [10] and is shown in Fig. 1 represented by the filled square.

In order to obtain the lower bound for finite K, one needs to have a two-time constraint that constrains the lower bound of $\overline{\Gamma}$. One way of finding a two-point constraint is shown in [10]. It is achieved as follows. To begin with, one finds the equation for δT by subtracting Eq. (2) from Eq. (1). Then, by multiplying the resulting equation by $u(t')$ at a different time t' and by taking both the ensemble average and the integration over x, one obtains the two-time constraint as

$$\mathcal{C}(\tau) = \frac{d}{d\tau}\overline{\Gamma}(\tau) - R^{-1}\overline{\Gamma''}(\tau) - U(\tau) = 0, \tag{9}$$

where $\tau \doteq t - t'$ and $U(\tau)$ is the autocorrelation func-

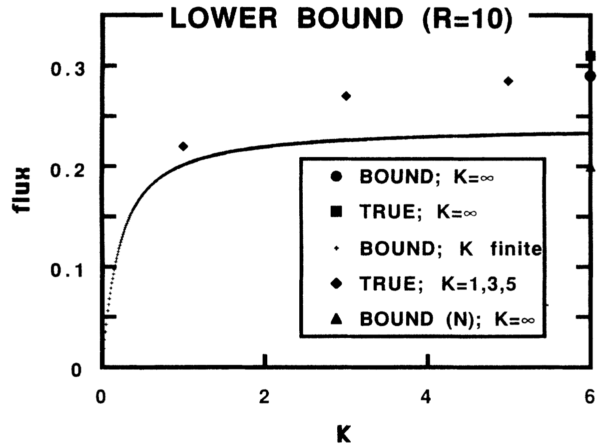

FIG. 1. Lower bounds for the flux in the case of $R = 10$. Circle denotes the lower bound without the two-point correlation ($K = \infty$); square represents the true value for $K = \infty$; curve is for the lower bound with two-point constraint; diamonds indicate the true values for K =1,3,5; triangle stands for the lower bound that is equivalent to the upper bound for the nonlinear term.

tion of u. Thus the new variational functional is $\mathcal{L}' = \mathcal{D} + \lambda\left[\bar{\Gamma}(0) - \mathcal{D} - \mathcal{N}\right] + \int d\tau\, \eta(\tau)\mathcal{C}(\tau)$ and η is the time-dependent Lagrange multiplier. To be more specific, we consider the case $U(\tau) = \exp(-|\tau|/K)$. The Euler-Lagrange equation becomes

$$-2R^{-1}\delta T'' + \lambda(u + 2R^{-1}\delta T'' - 2Ru\Delta\Gamma) - (\dot{\eta}\circ u)\left[1 + R^{-1}\delta''(\bar{x} - x)\,|_{\bar{x}=0}^{1}\right] = 0, \quad (10)$$

where $\circ$ means the convolution and the dot denotes differentiation with respect to time. Upon multiplying Eq. (10) by $\delta T(x,t)$ and by $u(t')$, respectively, one finds the multipliers in terms of the field variables as

$$\bar{\Gamma}\lambda = 2R^{-1}\overline{|\delta T'|^2} + (\eta\circ U)(0), \quad (11a)$$

$$(\dot{\eta}\circ U)(\tau) = -2(1-\lambda)\dot{\bar{\Gamma}} + (2-\lambda)U(\tau). \quad (11b)$$

Upon eliminating η by using Eq. (11b) one can reduce Eq. (10) to the form

$$\dot{\bar{\Gamma}}(x,\tau) - R^{-1}\Gamma''(x,\tau) + \left[\frac{\lambda R}{(\lambda-1)}\Delta\Gamma(x,0) - 1\right]U(\tau) = 0. \quad (12)$$

By using the method of separation of variables $[\Gamma(x,\tau) = \gamma(x)U(\tau)]$ one can reduce Eq. (12) to an ordinary second-order differential equation in x,

$$-\gamma'' + \frac{\lambda R^2}{(\lambda-1)}\gamma = R^2\left\{\bar{\gamma}\left[\frac{\lambda}{(\lambda-1)} - (RK)^{-1}\right] + R^{-1}\right\}. \quad (13)$$

It is relatively easy to solve Eq. (13) in conjunction with Eq. (11). The lower bound in terms of K for $R = 10$ is plotted in Fig. 1 along with the true values (represented by the filled diamonds) from the numerical computation for K =1,3,5 in [11]. One can note from Fig. 1 that in the case of infinite K the two lower bounds are different. This is because, in the limit of infinite K, Eq. (9) still is an additional condition on top of the energy balance Eq. (5) constraining the dissipation. This can be seen by the fact that, in Eq. (13), $\overline{\gamma''} = -R$, i.e., $\gamma'(1) = -\gamma'(0) = -R/2$ while, in Eq. (6), λ should be equal to 2 in order to satisfy $\overline{\gamma''} = -R$, which cannot happen because λ is always less than 2 as can be checked in Eq. (6). Thus, as expected, the inclusion of more constraints improves the upper bound for $\mathcal{D}$, which, though, is worsening the lower bound of the flux.

For completeness, we go on to compute the upper bound for $\mathcal{N}$ under the energy balance Eq. (5) but without the two-time constraint Eq. (9). The resulting Euler-Lagrange equation is

$$\Gamma'' + R(\lambda^{-1} - R)\Delta\Gamma + R/2 = 0, \quad (14)$$

where $\lambda = 2\overline{\Delta\Gamma^2}/\bar{\Gamma}$. The results are the following. For $R = 10$ the lower bound is 0.2 and for $R = 50$ it is 0.30, and the former result is shown in Fig. 1 as the filled triangle. The lower bound for $\bar{\Gamma}$ is lower in the case of maximizing $\mathcal{N}$ than in the case of maximizing $\mathcal{D}$. Thus, for this model, it is a better choice to maximize the dissipation instead of the nonlinear term in the energy production.

In the present passive model, it is shown that one can predict lower bounds reasonably well by employing the variational principle which is constrained by simple physical laws of conservation of the energy and the flux. It can be concluded that along with the upper bounds obtained in a similar fashion one can bound the flux fairly well. It can be argued as well that, although the present model is primitive, there exists enough evidence from the previous studies on upper bounds that, even for self-consistent problems, this method will be useful. One thing typically different is that, in self-consistent problems, there is no imposed parameter like K of the present model. Rather, "K" is to be determined in terms of the parameters like R. So, one may not need the two-time constraint as in this work since an additional constraint may make the lower bounds worse. Basically, the energy balance is just enough to find a reasonable lower bound. It is natural to consider lower bounds for fluxes in physical problems and this is now under study for plasma turbulence. It should be noted that the bounding method followed in this work does not replace the dynamical closure theories like the direct-interaction approximation (DIA). It does not predict the temporal behaviors since it is a method suitable for the stationary state and it is impractical to incorporate large numbers of constraints. However, its merit is to be found in the rigor of providing the bounds for the transport. It is shown in [10] that, as far as the absolute error goes, the DIA is better than the bounding method. It is also shown that the result of the DIA can be either larger or smaller than the true value while the upper bound stays always larger than and close to the true one.

ACKNOWLEDGMENT

The author is grateful for the hospitality of the Institute for Fusion Studies at the University of Texas at Austin, where part of the work was performed.

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